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# Fatigue of continuous fibre composites

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Fatigue curves for composites are derived from those of the components, with as little arbitrariness as possible. Even with this restriction, the expected fatigue behaviour of the composites turns out to be of a rather diverse nature, depending on modulus ratio, volume fraction, and residual stress. The results concern pulsating tension as well as fluctuating load at an arbitrary mean stress. By comparison with available experimental data, our results lead to a better understanding of composite behaviour.

## 1. Introduction

Fatigue strength is one of the most important mechanical properties of a material. Since its experimental determination is rather time-consuming, detailed investigations as to the expected outcome of experiments or to the interpretation of the results may be justified. We will not go into details of the fatigue of metals or other materials but instead will try to answer the question of what kind of fatigue curve should be expected if the fatigue curves of the components of a composite material are known. Damage of composites is a rather complex phenomenon, including matrix cracks, interface debonding, fibre cracks, pull-out, local yield, fibres bridging the gap, etc. We will neither investigate those mechanisms nor go to the other extreme of constructing composite fatigue curves according to the rule of mixtures, which would be a clearly over-simplified approach.

We are considering the behaviour of a composite material consisting of continuous aligned fibres embedded in a matrix. The components are assumed to be ideally elastic, i.e. redistribution of stress by creep is not considered. Differences between Poisson's ratios of the components are neglected. The components are supposed to behave inside the composite in the same way as they would do. By comparison with experiment, the results obtained for the idealized system should enable us to draw conclusions concerning complex phenomena occurring in real systems. For simplicity, the results are derived initially without considering residual stress. This is taken into account later.

Symbols and subscripts are consistent with those of a former paper [1]. $\sigma$, $\epsilon$ and $E$ are axial stress, strain, and Young's modulus respectively of the composite; M, F and K as first subscripts refer to the matrix, fibre, or any component respectively. Subscript f represents failure due to fatigue, while subscripts fM and fF represent composite damage due to failure of matrix or fibre, respectively. $v$ is the volume fraction of fibres, and $N$ the number of cycles.

The fatigue curves of the components are written as $\sigma_{\mathrm{Mf}}(N)$ and $\sigma_{\mathrm{Ff}}(N)$, while those of the composite are written $\sigma_{\mathrm{f1}}(N)$ and $\sigma_{\mathrm{f2}}(N)$, referring to failure of one or two components, respectively. It will be necessary to work mainly in terms of the reverse functions, $N_{\mathrm{Mf}}(\sigma_{\mathrm{M}})$, $N_{\mathrm{Ff}}(\sigma_{\mathrm{F}})$, $N_{\mathrm{f1}}(\sigma)$, and $N_{\mathrm{f2}}(\sigma)$. The subscripts should be thought of as belonging to the curve rather than to the symbol they are attached to, lest formal objections arise against arbitrarily changing from $\sigma_{\mathrm{f2}}(N)$ to $N_{\mathrm{f2}}(\sigma)$, for instance, which is necessary in the course of our reasoning.

## 2. Fatigue under constant strain amplitude

The axial load carried by the elastic components of a continuous fibre composite at strain $\epsilon$ in the absence of residual stress is given by

$$\sigma_{\mathrm{M}} = \epsilon_{\mathrm{M}} E_{\mathrm{M}}, \quad \sigma_{\mathrm{F}} = \epsilon_{\mathrm{F}} E_{\mathrm{F}}, \quad \epsilon_{\mathrm{M}} = \epsilon_{\mathrm{F}} = \epsilon$$
$$\sigma = v\sigma_{\mathrm{F}} + (1-v)\sigma_{\mathrm{M}} = \epsilon E \tag{1}$$
$$E = vE_{\mathrm{F}} + (1-v)E_{\mathrm{M}}$$

Failure of the fibres supposedly occurs at the number of cycles, $N$, at which the fatigue curve $\sigma_{\mathrm{Ff}}(N)$

has dropped to the value of the actual fibre stress $\sigma_F$. This condition, with Equation 1, leads to

$$\sigma_{Ff}(N) = \sigma_F = \epsilon E_F = \sigma E_F/E. \qquad (2)$$

$\sigma$ in this formula has the meaning of $\sigma_{fF}$, the stress at which the composite suffers damage due to fibre failure;

$$\sigma_{fF} = \sigma_{Ff}(N)E/E_F. \qquad (3a)$$

Similar reasoning gives a corresponding formula for the case of early matrix damage, the fibres being unaffected;

$$\sigma_{fM} = \sigma_{Mf}(N)E/E_M. \qquad (3b)$$

Using these formulae, one need not know in advance which of the two components would fail first. Such information can be obtained by comparison of $\sigma_{fF}$ of Equation 3a with $\sigma_{fM}$ of Equation 3b.

$$\sigma_{fM} \lessgtr \sigma_{fF} \quad \leftrightarrow \quad \frac{\sigma_{Mf}}{E_M} \lessgtr \frac{\sigma_{Ff}}{E_F} \quad \leftrightarrow \quad \left.\begin{matrix}\text{matrix}\\ \text{fibre}\end{matrix}\right\} \begin{matrix}\text{fails}\\ \text{first}\end{matrix} \qquad (3c)$$

The composite stress decreases by the factor $(1-v)E_M/E$ as soon as the fibres fail, since the strain amplitude is kept constant. In the $(\sigma, N)$-plane of the fatigue curves, the sample would then cease moving parallel to the abscissa as $N$ increases and would "jump" to the decreased ordinate. Obtaining information from such plots would not be easy, and so for convenience we keep each sample moving at its ordinate even after failure of one component. Thus Equations 3a and b may be regarded as true, regardless of the sequence by which the components fail.

Since no special assumptions are made as to the shape of the fatigue curves of the components, our results apply to any monotonic component fatigue curve. For simplicity we use simple component fatigue curves that are composed of two straight sections in the logarithmic representation, as is often observed experimentally, with reasonable accuracy, if low-cycle fatigue is ignored (Fig. 1).

As may be seen from Fig. 1, the character of composite failure changes with increasing stress. Below 1.6 (arbitrary units) fatigue failure does not occur at all. Between 1.6 and 2.1, the matrix fails, while the reinforcement does not, and between 2.1 and 3.9 the reinforcement fails first, the failure of the matrix being more or less delayed. Above 3.9 the sequence is reversed.

It will be useful to reason in terms of $\sigma_{f1}$ and $\sigma_{f2}$;

$$\sigma_{f1}(N) = \min \begin{cases} \sigma_{fM}(N) \\ \sigma_{fF}(N) \end{cases} \qquad (4a)$$

$$\sigma_{f2}(N) = \max \begin{cases} \sigma_{fM}(N) \\ \sigma_{fF}(N). \end{cases} \qquad (4b)$$

$\sigma_{f1}(N)$ is the stress at which composite damage occurs by failure of one component at $N$ cycles, and $\sigma_{f2}(N)$ is the stress which causes complete failure of both components at $N$ cycles (Figs. 2 and 3).

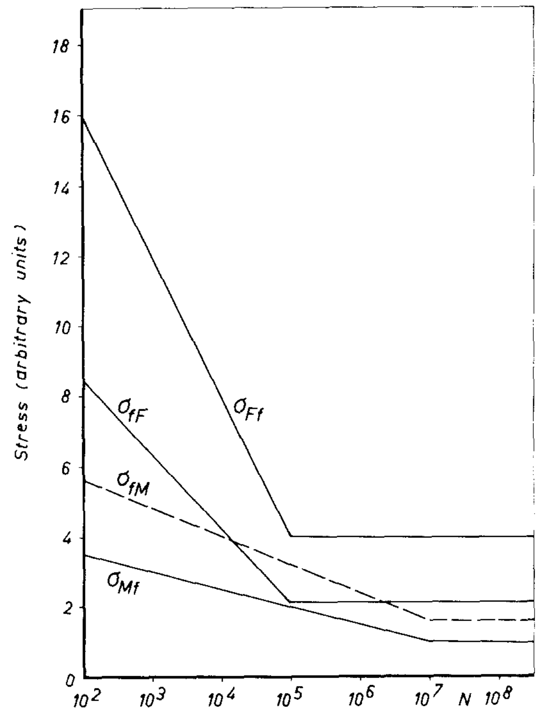


*Figure 1* $\sigma_{Ff}$ and $\sigma_{Mf}$ are arbitrarily chosen fatigue curves of the components at pulsating tension. $\sigma_{fF}$ and $\sigma_{fM}$ are the derived fatigue curves of the composites with $v = 0.3$ and $E_F/E_M = 3$. The dashed line shows fibre and matrix failure at pulsating tension, with strain amplitude kept constant.

## 3. Fatigue under constant stress amplitude

Instead of the strain amplitude, the stress amplitude is often kept constant during the fatigue test. The situation is identical to that of Section 2 as long as the deformation is purely elastic. Thus damage due to failure of one component, which occurs at $\sigma_{f1}$, is properly described by Equation 4a. The number of cycles to failure of the remaining component requires more detailed consideration.

For several reasons it becomes easier to work

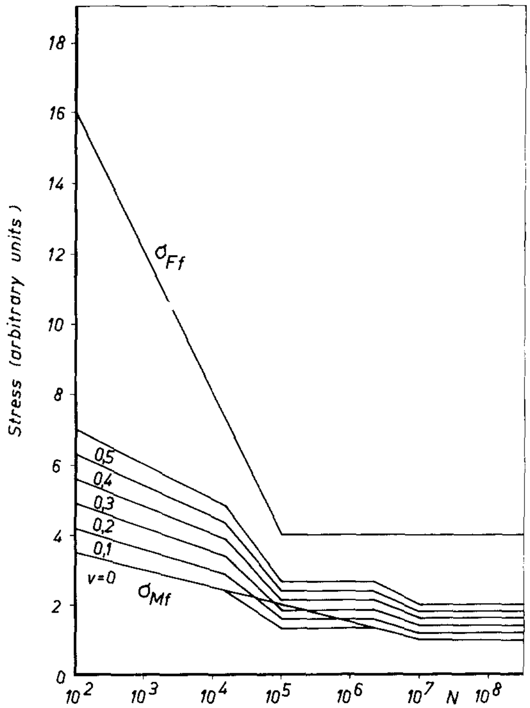


*Figure 2* $\sigma_{\mathrm{Ff}}$ and $\sigma_{\mathrm{Mf}}$ as in Fig. 1. The family of curves are the derived fatigue curves $\sigma_{\mathrm{f}_1}$ of the composite with $E_{\mathrm{F}}/E_{\mathrm{M}} = 3$, showing failure of one component at pulsating tension with strain amplitude kept constant.

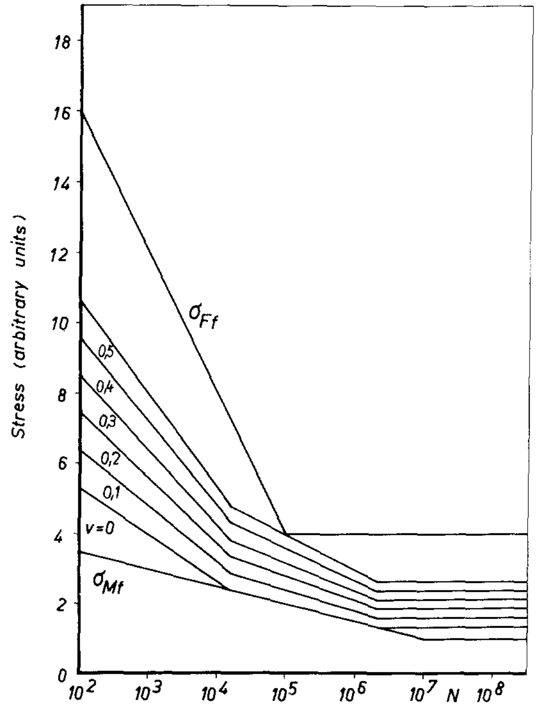


*Figure 3* $\sigma_{\mathrm{Ff}}$ and $\sigma_{\mathrm{Mf}}$ as in Fig. 2. The family of curves are the derived fatigue curves $\sigma_{\mathrm{f}_2}$ of the composite with $E_{\mathrm{F}}/E_{\mathrm{M}} = 3$, showing failure of two components at pulsating tension, with strain amplitude kept constant.

with the reverse functions, such as $N_{\mathrm{Ff}}(\sigma_{\mathrm{F}})$, i.e. the number of cycles to failure as a function of the applied load, the latter being treated as an independent variable. The subscripts on the stress variables are to distinguish them from the composite stress, which goes without a subscript. Making use of $\epsilon_{\mathrm{F}} = \epsilon_{\mathrm{M}}$, we may express $\sigma_{\mathrm{F}}$ and $\sigma_{\mathrm{M}}$ by $\sigma$;

$$\sigma_{\mathrm{F}} = \sigma E_{\mathrm{F}}/E, \quad \sigma_{\mathrm{M}} = \sigma E_{\mathrm{M}}/E. \tag{5}$$

Then the number of cycles to failure of one component is given by

$$N_{\mathrm{f}1}(\sigma) = \min \left\{ \begin{matrix} N_{\mathrm{Mf}}(\sigma E_{\mathrm{M}}/E) & \text{matrix} \\ & \rightarrow \\ N_{\mathrm{Ff}}(\sigma E_{\mathrm{F}}/E) & \text{fibre} \end{matrix} \right\} \text{fails first} \tag{6}$$

As soon as one component has failed, the remaining one has to carry the whole load. The problem of calculating its endurance cannot be solved by considering the conventional fatigue curves only, for they do not contain information concerning the number of cycles to failure in such a test where the load amplitude is not kept constant. Thus we have to invent component fatigue functions of the type $N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K}1}, \sigma_{\mathrm{K}})$. This represents the number of cycles to failure of a material K (standing for M or F) under the load $\sigma_{\mathrm{K}}$ after having been loaded with the smaller load $\sigma_{\mathrm{K}1}$ up to $N_1$ cycles. Using these functions we may write the number of cycles to failure of both components as,

$$N_{\mathrm{f}2}(\sigma) = \begin{cases} N_{\mathrm{Ff}}(N_{\mathrm{f}1}, \sigma E_{\mathrm{F}}/E, \sigma/v) & \text{if the matrix fails first} \\ N_{\mathrm{Mf}}(N_{\mathrm{f}1}, \sigma E_{\mathrm{M}}/E, \sigma/(1-v)) & \text{if the fibre fails first} \end{cases} \tag{7}$$

where the formal parameter $N_1$ of $N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K}1}, \sigma_{\mathrm{K}})$ has been identified with $N_{\mathrm{f}1}$ of Equation 6, $\sigma_{\mathrm{K}1}$ with $\sigma_{\mathrm{F}}$ or $\sigma_{\mathrm{M}}$ of Equation 5, and $\sigma_{\mathrm{K}}$ with the increased stress on the remaining component.

Fortunately it is not necessary to determine these functions experimentally, because narrow bounds expressed in terms of ordinary fatigue functions can be found for them. We use these auxiliary fatigue functions of more than one parameter in intermediates stages of working only, and represent the results in terms of the ordinary functions. In order to derive the bounds, some inherent properties of the functions are considered which are independent of the material. Several relations follow from the definition of the functions given above;

$$\begin{aligned} N_{\mathrm{Kf}}(0, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}) &= N_{\mathrm{Kf}}(\sigma_{\mathrm{K}}) \\ N_{\mathrm{Kf}}(N_1, 0, \sigma_{\mathrm{K}}) &= N_{\mathrm{Kf}}(\sigma_{\mathrm{K}}) + N_1 \\ N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K}}, \sigma_{\mathrm{K}}) &= N_{\mathrm{Kf}}(\sigma_{\mathrm{K}}). \end{aligned} \tag{8}$$

$N_1$ cannot exceed $N_{\mathrm{Kf}}$ by definition, hence

$$N_1 < N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}). \tag{9a}$$

On the assumption that the material behaves "reasonably", i.e. every increase in load at any cycle drives the material slightly further towards final failure, the following inequality may be set up;

$$\begin{aligned} &N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K}}, \sigma_{\mathrm{K}}) < N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}) \\ &< N_{\mathrm{Kf}}(N_1, 0, \sigma_{\mathrm{K}}), \end{aligned} \tag{9b}$$

which means that the total number of cycles to failure is larger the smaller the load during the first $N_1$ cycles out of the total number. Any reduction in load will increase $N_{\mathrm{Kf}}$, hence

$$N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}) < N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K1}}) \tag{9c}$$

Making use of Equation 8 and by a little rearrangement, Inequality 9 is reduced to

$$N_{\mathrm{Kf}}(\sigma_{\mathrm{K}}) < N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}) < N_{\mathrm{Kf}}(\sigma_{\mathrm{K1}})$$

and

$$N_1 < N_{\mathrm{Kf}}(N_1, \sigma_{\mathrm{K1}}, \sigma_{\mathrm{K}}) < N_1 + N_{\mathrm{Kf}}(\sigma_{\mathrm{K1}}) \tag{10}$$

These relations provide bounds for $N_{\mathrm{f2}}(\sigma)$, which are much narrower than experimental scatter in almost all cases of practical interest. If applied to Equation 7, they lead to the results below:

if the matrix fails first then

$$N_{\mathrm{Ff}}(\sigma/v) \leqslant N_{\mathrm{f2}}(\sigma) \leqslant N_{\mathrm{Ff}}(\sigma E_{\mathrm{F}}/E) \quad \text{and}$$

$$N_{\mathrm{f1}}(\sigma) \leqslant N_{\mathrm{f2}}(\sigma) \leqslant N_{\mathrm{f1}}(\sigma) + N_{\mathrm{Ff}}(\sigma/v)$$

if the fibre fails first then

$$N_{\mathrm{Mf}}(\sigma/(1-v)) \leqslant N_{\mathrm{f2}}(\sigma) \leqslant N_{\mathrm{Mf}}(\sigma E_{\mathrm{M}}/E) \quad \text{and}$$

$$N_{\mathrm{f1}}(\sigma) \leqslant N_{\mathrm{f2}}(\sigma) \leqslant N_{\mathrm{f1}}(\sigma) + N_{\mathrm{Mf}}(\sigma/(1-v)) \tag{11}$$

$$0 \leqslant N_{\mathrm{f2}}(\sigma) - N_{\mathrm{f1}}(\sigma) \leqslant \begin{cases} N_{\mathrm{Ff}}(\sigma/v) & \text{if the matrix fails first} \\ N_{\mathrm{Mf}}(\sigma/(1-v)) & \text{if the fibre fails first.} \end{cases} \tag{12}$$

Which of the bounds of $N_{\mathrm{f2}}(\sigma)$ will be relevant depends on the parameters of the special system in question as well as on $\sigma$.

Equation 11 allows us to write down bounds for $(N_{\mathrm{f2}} - N_{\mathrm{f1}})$, which are a useful means for assessing results because they are rather narrow in many cases, as in the model composite material that Figs. 1 to 3 are based on (see Inequality 12). In Fig. 2, where $N_{\mathrm{f1}}$ is shown, $N_{\mathrm{f2}}$ and $N_{\mathrm{f1}}$ would practically coincide at small $v$ up to $v = 0.3$. Visible differences would arise at about $v = 0.4$

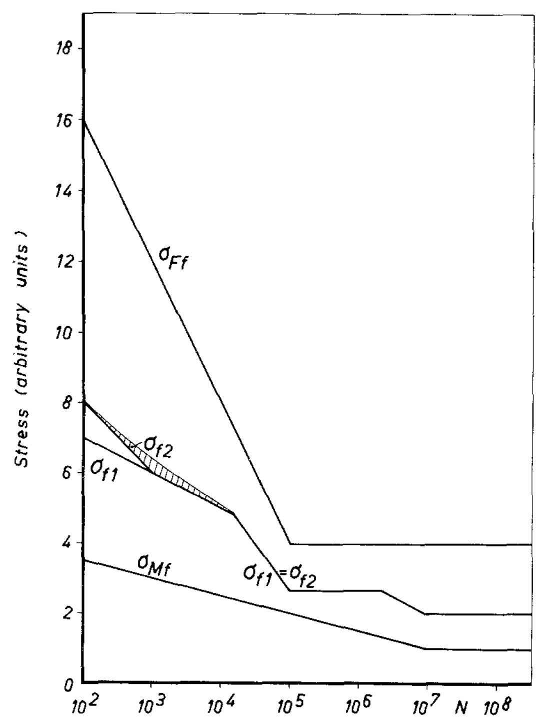


*Figure 4* $\sigma_{\mathrm{Ff}}$ and $\sigma_{\mathrm{Mf}}$ as in Fig. 1. $\sigma_{\mathrm{f_1}}$ and $\sigma_{\mathrm{f_2}}$ are the derived fatigue curves of the composite with $E_{\mathrm{F}}/E_{\mathrm{M}} = 3$ and $v = 0.5$, showing failure of one or two components, respectively. The stress amplitude at pulsating tension is kept constant. The shaded region indicates the uncertainty of $\sigma_{\mathrm{f_2}}$ inherent in our result.

and above. (The statements about the shape of the curves at different values of $v$ apply to the special case that is chosen here for illustration. Different component curves $\sigma_{Mf}$ and $\sigma_{Ff}$ could alter the situation, as may be seen in connection with the discussion of experimental data in Section 4. Fig. 3 applies to the case of constant strain amplitude only and therefore is not applicable to the situation dealt with in this section).

The situation described above is shown in Fig. 4 for $v = 0.5$. The shaded region represents the uncertainty in $N_{f2}(\sigma)$, or $\sigma_{f2}(N)$, arising from the approximate character of our formulae. If necessary, the uncertainty could be reduced by more involved equations. We feel, however, that there is no need for greater subtlety in the theoretical approach, since some uncertainty of experimental fatigue curves as well as inevitable imperfections inherent in our theory due to neglect of more involved synergetic effects have to be anticipated. This does not apply to residual stress, the effect of which will be taken into account in Section 6.

## 4. Discussion of experimental data

Experimental results on fatigue of composite materials are not abundant, and most of the existing composite fatigue curves have been measured without measuring those of the components at the same time. Thus it is not possible, at present, to derive definitely from experimental data to what extent composites really behave in the simple manner upon which our model is based.

Varshavski and Tamayo [2] have investigated a steel wire/aluminium alloy composite. With the asymptotic fatigue data of the components, steel $366\,N\,mm^{-2}$ and Al alloy $126\,N\,mm^{-2}$, with Young's moduli of $205\,000\,N\,mm^{-1}$ and $72\,000\,N\,mm^{-2}$, respectively, our Equation 4a* in connection with Relations 3 leads to the result that the matrix will fail finally if the composite stress exceeds $\sigma_{f1} = 183\,N\,mm^{-2}$. This coincides remarkably well with the experimental value of $177\,N\,mm^{-2}$ reported in [2]. Since after matrix failure, the fibres covering only 25% of the cross-section would be stressed at twice the value of their fatigue strength, they will break immediately afterwards. Thus we come to the conclusion that $\sigma_{f2} = \sigma_{f1}$, without, in this case, having to worry about formulae in the $N(\sigma)$-representation, such as Inequality 11 which is somewhat tedious if only the asymptotic behaviour is of interest.

Our results seem to indicate that that composite material from Harvey Aluminium behaves as predicted by our theory based on the least arbitrary assumptions, as far as fatigue is concerned. Thus there is no need to explain fatigue damage of the material by debonding of interfaces, and the conclusion in [2] that fatigue properties might be improved by improving the fibre–matrix bond is not convincing.

The tensile strength data of the components and the composite of [2] are incompatible. Perhaps the composite tensile strength of $1200\,N\,mm^{-2}$ is erroneous; one would expect less than half as much. The suspicion aroused by this figure is inevitably transferred to others connected with it, such as the endurance ratio. Another paper on fatigue by Varshavski [3] has been reviewed in [1].

Experimental data obtained by Friedrich and Busalov [4] seem to fit well into our scheme, though some restrictions have to be considered, as will be discussed later. Fig. 5 shows fatigue curves of a composite, the components of which differ

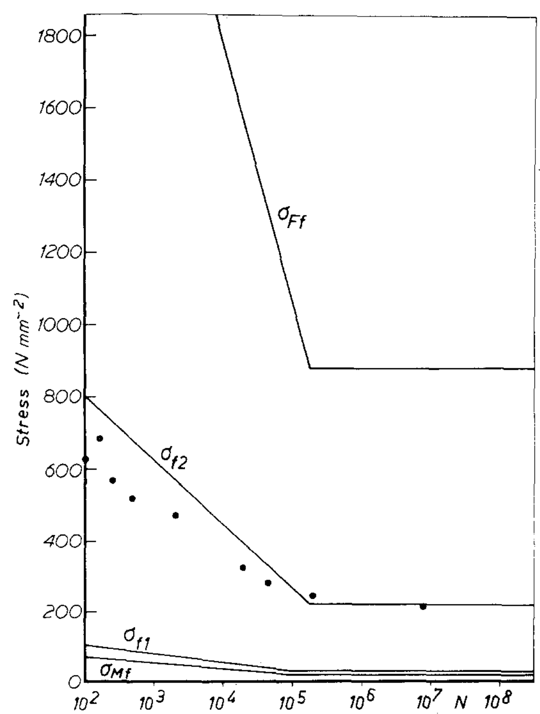


*Figure 5* $\sigma_{Ff}$ is the rotary bending fatigue curve of Mo-maraging steel wires as in [4] (teflon channel method). $\sigma_{Mf}$ is the fatigue curve of the Al alloy matrix used in [4], while $\sigma_{f1}$ and $\sigma_{f2}$ are the derived fatigue curves of the Al alloy/steel wire composite, with $v = 0.25$. Dots represent pulsating tension fatigue data by [4].

*Which also holds under constant stress amplitude.

extremely in their strength. As a result, the curves $\sigma_{f1}(N)$ and $\sigma_{f2}(N)$ are placed far apart. It is clearly seen in our graphical representation that the experimental data most probably belong to $\sigma_{f2}$, while their large distance from the curve $\sigma_{f1}$ indicates that the matrix may have been damaged at an early stage of the test. This would mean that the experimental data of Fig. 5 may represent not the strength of a compact composite material but that of a bunch of more or less loosely connected wires. The same applies to Fig. 6.

The good correspondence between their experimental results and our theory has to be looked at with suspicion. This doubt arises from the suspiciously good performance of the Mo-maraging steel wires, $\sigma_{Ff}(N)$. The 0.15 mm diameter wires had been tested with a rotary-bending method described in [5]. In this test the wires rotate within a curved channel inside a piece of teflon. Putting aside the question of how to relate the rotary-bending fatigue strength to the pulsating tension fatigue strength, the problem remains that the curved channel method might give much higher endurances than the ordinary rotary-bending method. The curvature of the wire is kept constant by the channel and any reduction of the wire cross-section by growing fatigue cracks reduces the bending moment at that site. The crack tip reaches the lower-stressed material inside the wire, where it may stop. With the ordinary rotary-bending fatigue test the bending moment is kept nearly constant, which leads to stress increase as soon as cracks begin to grow, causing the latter to accelerate. Thus the ordinary rotary-bending fatigue strength of this steel wire may be markedly lower than the value reported in [4]. For comparison with pulsating tension fatigue data of the composite, the pulsating tension fatigue strength of the wire should be approximately calculated from the disputed value using some empirical relation between them, like Goodman's law [6]. Thus it seems that after two necessary corrections, which roughly compensate each other, the true steel wire fatigue curve at pulsating tension may indeed come close to that curve reported by [4]. This may explain the good correspondence between theory and experiment in Figs. 5 and 6. In addition to these comments on [4], it has been found with the aid of our theory that one of their sets of experimental results has to be rejected. Despite this it has unfortunately found its way into a paper on composites [7].

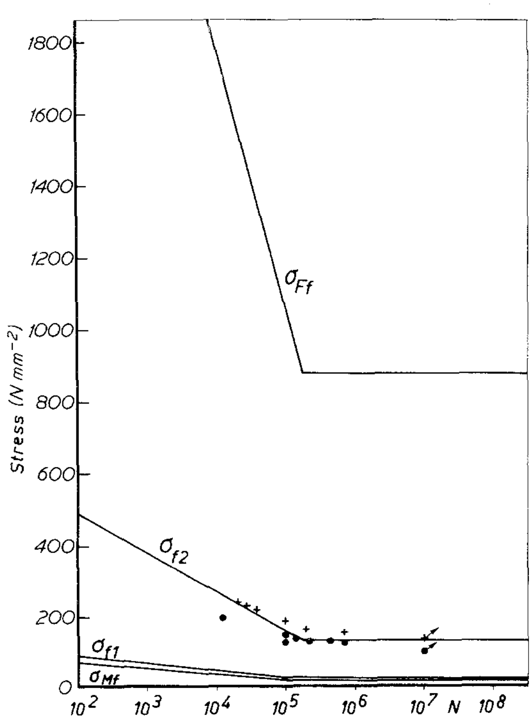


*Figure 6* As in Fig. 5, but with $v = 0.15$. Dots and crosses represent two sets of pulsating tension fatigue data by [4].

Further experimental data as well as theoretical approaches will be discussed in connection with the results of the following sections.

## 5. Fatigue under the condition of arbitrary mean stress

In the preceding sections fatigue has been investigated only in the case of tensile stress changing between zero and the stress amplitude $\sigma$ or $\sigma_K$, which means that the mean stress is half as much. Often it is desirable, however, to obtain information on composite behaviour under the more general stress condition of a non-zero lower stress. Then fatigue is represented by a relation between $N$, mean stress, and alternating stress. The following additional symbols are used; superscript m represents the mean stress, superscript a represents the alternating stress, and subscript u represents ultimate stress.

The component fatigue functions may be written in the form $N_{Kf}(\sigma_K^m, \sigma_K^a)$, denoting the number of cycles to failure of the material which is cyclically stressed with $\sigma_K^a$ superposed on the mean stress $\sigma_K^m$. The stress condition discussed in the preceding sections, for comparison, is characterized by the restriction $\sigma_K = 2\sigma_K^a = 2\sigma_K^m$. The factor 2 enters,

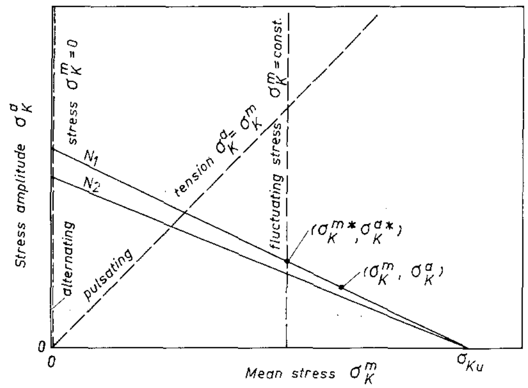


*Figure 7* Diagram illustrating the meaning of several terms as well as Goodman's Law.

since $\sigma_K^m$ is regarded as reference zero for the alternating stress. The relation between $N_{Kf}(\sigma_K^m, \sigma_K^a)$ and our pulsating stress functions of the preceding sections, $N_{Kf}(\sigma_K)$, follows from their definition:

$$N_{Kf}(\sigma_K) = N_{Kf}(\sigma_K/2, \sigma_K/2) \qquad (13)$$

(Since the differently defined functions $N_{Kf}$ can be distinguished by their number of variables, no other means of specification has been chosen.) It would be convenient to express $N_{Kf}(\sigma_K^m, \sigma_K^a)$ by $N_{Kf}(\sigma_K)$ in the entire range of the variables $\sigma_K^m$ and $\sigma_K^a$. The clue is provided by empirical laws such as the (modified) Goodman law, which holds with sufficient accuracy for many materials [6].

Goodman's law may be stated in the following way: If $N_{Kf}$ is kept constant, $N_{Kf}(\sigma_K^m, \sigma_K^a)$ produces straight lines in the $(\sigma_K^m, \sigma_K^a)$-plane which intersect the $\sigma_K^m$-axis at $\sigma_{Ku}$ (Fig. 7). That means that any pair of variables $(\sigma_K^m, \sigma_K^a)$ belonging to that line is equivalent with respect to the number of cycles to failure. If $N_{Kf}$ has been determined at a special loading regime $(\sigma_K^{m*}, \sigma_K^{a*})$, then the same number $N_{Kf}$ has to be expected for any $(\sigma_K^m, \sigma_K^a)$ which satisfies the linear equation

$$\frac{\sigma_K^{a*}}{\sigma_{Ku} - \sigma_K^{m*}} = \frac{\sigma_K^a}{\sigma_{Ku} - \sigma_K^m} . \qquad (14)$$

This relation is evident from Fig. 7. (The subscript K is to remind the reader of the fact that in this section we have not been concerned with the composite so far, but with the components.) In particular, with the aid of Goodman's law, it is possible to substitute any fluctuating stress regime by an equivalent regime of pulsating tension, $\sigma_K^a = \sigma_K^m = \sigma_K/2$. By doing so, we obtain from Equation 14,

$$\sigma_K = \frac{2\sigma_{Ku}\,\sigma_K^{a*}}{\sigma_{Ku} - \sigma_K^{m*} + \sigma_K^{a*}} . \qquad (15)$$

(The asterisks may be omitted if not needed to distinguish between variables.) Hence, we may write

$$N_{Kf}(\sigma_K^m, \sigma_K^a) = N_{Kf}\left(\frac{2\sigma_{Ku}\,\sigma_K^a}{\sigma_{Ku} + \sigma_K^a - \sigma_K^m}\right) \qquad (16)$$

which indicates how information on fatigue under the more general condition of fluctuating load may be derived from a pulsating tension fatigue curve.

In the absence of yield, the component stresses are related to the composite stress by

$$\sigma_K^m = \sigma^m E_K/E_I; \quad \sigma_K^a = \sigma^a E_K/E_I. \qquad (17)$$

Now the question arises as to which of the components will fail first, initiating damage of the composite. It will be that of the lower number $N_{Kf}$ at a given composite stress,

$$N_{f1}(\sigma^m, \sigma^a) = \min \begin{cases} N_{Mf}(\sigma_M^m, \sigma_M^a) \\ N_{Ff}(\sigma_F^m, \sigma_F^a) . \end{cases} \qquad (18)$$

Taking into account Equations 16 and 17, we obtain $N_{f1}(\sigma^m, \sigma^a)$, expressed by the pulsating tension functions of the components, as given by Equation 22 at the end of this section.

For the purpose of calculating $N_{f2}$, which is the number of cycles to failure of the remaining component, information on the fatigue of the components would be required in the form of functions of five variables, $N_{Kf}(N_{f1}, \sigma_{K1}^m, \sigma_{K1}^a, \sigma_K^m, \sigma_K^a)$. Again the many-parameter fatigue functions serve only as an auxiliary means for deriving boundaries or approximate solutions. The final failure of the composite is determined by

$$N_{f2} = \begin{cases} N_{Ff}(N_{f1}, \sigma_{F1}^m, \sigma_{F1}^a, \sigma_F^m, \sigma_F^a) & \text{if the matrix fails first} \\ N_{Mf}(N_{f1}, \sigma_{M1}^m, \sigma_{M1}^a, \sigma_M^m, \sigma_M^a) & \text{if the fibre fails first} \end{cases} \qquad (19)$$

Goodman's law reduces the five-parameter functions to three-parameter functions, the arithmetic of which was worked out in the preceding section.

$$N_{\mathrm{Kf}}(N_{\mathrm{f1}}, \sigma_{\mathrm{K1}}^{\mathrm{m}}, \sigma_{\mathrm{K1}}^{\mathrm{a}}, \sigma_{\mathrm{K}}^{\mathrm{m}}, \sigma_{\mathrm{K}}^{\mathrm{a}}) \tag{20}$$

$$= N_{\mathrm{Kf}}\left(N_{\mathrm{f1}}, \frac{2\sigma_{\mathrm{Ku}}\sigma_{\mathrm{K1}}^{\mathrm{a}}}{\sigma_{\mathrm{Ku}} + \sigma_{\mathrm{K1}}^{\mathrm{a}} - \sigma_{\mathrm{K1}}^{\mathrm{m}}}, \frac{2\sigma_{\mathrm{Ku}}\sigma_{\mathrm{K}}^{\mathrm{a}}}{\sigma_{\mathrm{Ku}} + \sigma_{\mathrm{K}}^{\mathrm{a}} - \sigma_{\mathrm{K}}^{\mathrm{m}}}\right).$$

With the component stresses replaced by composite stresses,

$$\sigma_{\mathrm{KI}}^{\mathrm{a,m}} = \sigma^{\mathrm{a,m}} E_{\mathrm{K}}/E_{\mathrm{I}} \tag{21}$$

$$\sigma_{\mathrm{F}}^{\mathrm{a,m}} = \sigma^{\mathrm{a,m}}/v, \quad \sigma_{\mathrm{M}}^{\mathrm{a,m}} = \sigma^{\mathrm{a,m}}/(1-v).$$

$N_{\mathrm{f2}}$ reads as Equation 23. Making use of Inequality 10, we obtain the Inequality 24. Note that Relations 22 and 24, despite their formidable appearance, serve as explicit expressions ready for straightforward computation of the desired numbers $N_{\mathrm{f1}}$ and $N_{\mathrm{f2}}$. Part of the information may be picked out to provide a preliminary survey of the result, as in Inequality 12.

The results mentioned in this section are given below, with the parameters $S_1$, $S_2$, $S_3$ and $S_4$ representing more complex functions of the other variables.

$$S_1 \equiv \frac{2\sigma_{\mathrm{Mu}}\sigma^{\mathrm{a}}E_{\mathrm{M}}}{\sigma_{\mathrm{Mu}}E + (\sigma^{\mathrm{a}} - \sigma^{\mathrm{m}})E_{\mathrm{M}}}, \quad S_3 \equiv \frac{2\sigma_{\mathrm{Mu}}\sigma^{\mathrm{a}}}{(1-v)\sigma_{\mathrm{Mu}} + \sigma^{\mathrm{a}} - \sigma^{\mathrm{m}}},$$

$$S_2 \equiv \frac{2\sigma_{\mathrm{Fu}}\sigma^{\mathrm{a}}E_{\mathrm{F}}}{\sigma_{\mathrm{Fu}}E + (\sigma^{\mathrm{a}} - \sigma^{\mathrm{m}})E_{\mathrm{F}}}, \quad S_4 \equiv \frac{2\sigma_{\mathrm{Fu}}\sigma^{\mathrm{a}}}{v\sigma_{\mathrm{Fu}} + \sigma^{\mathrm{a}} - \sigma^{\mathrm{m}}}.$$

$$N_{\mathrm{f1}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) = \min \begin{cases} N_{\mathrm{Mf}}(S_1) & \rightarrow \text{ matrix fails first} \\ N_{\mathrm{Ff}}(S_2) & \phantom{\rightarrow} \text{ fibre fails first} \end{cases} \tag{22}$$

$$N_{\mathrm{f2}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) = \begin{cases} N_{\mathrm{Ff}}(N_{\mathrm{f1}}, S_2, S_4) & \text{if the matrix fails first} \\ N_{\mathrm{Mf}}(N_{\mathrm{f1}}, S_1, S_3) & \text{if the fibre fails first} \end{cases} \tag{23}$$

(This is not a final result)

If the matrix fails first then $N_{\mathrm{Ff}}(S_4) \leqslant N_{\mathrm{f2}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{Ff}}(S_2)$.

and $N_{\mathrm{f1}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{f2}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{f1}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) + N_{\mathrm{Ff}}(S_4)$.

If the fibre fails first then $N_{\mathrm{Mf}}(S_3) \leqslant N_{\mathrm{f2}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{Mf}}(S_1)$

and $N_{\mathrm{f1}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{f2}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) \leqslant N_{\mathrm{f1}}(\sigma^{\mathrm{m}}, \sigma^{\mathrm{a}}) + N_{\mathrm{Mf}}(S_3)$.

(24)

## 6. The effect of residual stress on fatigue

The existence of residual stress must always be anticipated in composite materials. It may enter during the process of production as a result of differential thermal contradiction or in connection with cold-working [1]. Since the residual forces on the components necessarily cancel due to the equilibrium of forces, only one parameter is needed to characterize the distribution of axial residual stress in the composite material consisting of two components. It is called $\sigma_0$, which is defined by

$$v\sigma_{\mathrm{F0}} = -(1-v)\sigma_{\mathrm{M0}} = \sigma_0. \tag{25}$$

Experimental evidence [1] as well as theoretical estimates lead to the conclusion that neglecting stress components other than axial, as has been done in our approach, provides a good approximation. For this reason, we consider axial residual stress only.

The stress acting on the components is simply residual stress plus stress due to external load,

$$\sigma_{\mathrm{F}} = \sigma E_{\mathrm{F}}/E_{\mathrm{I}} + \sigma_{\mathrm{F0}} \quad \sigma_{\mathrm{M}} = \sigma E_{\mathrm{M}}/E_{\mathrm{I}} + \sigma_{\mathrm{M0}} \tag{26}$$

Thus residual stress shifts the mean stresses $\sigma_{\mathrm{F}}^{\mathrm{m}}$

and $\sigma_M^m$, leaving the amplitudes of alternating stress, $\sigma_F^a$ and $\sigma_M^a$, unchanged.

$$\sigma_F^m = \sigma^m E_F/E_I + \sigma_0/v \qquad \sigma_F^a = \sigma^a E_F/E_I$$
$$\sigma_M^m = \sigma^m E_M/E_I - \sigma_0/(1-v) \qquad \sigma_M^a = \sigma^a E_M/E_I \qquad (27)$$

Using these expressions instead of Equations 17, we obtain Equation 30, instead of Equation 22.

Writing down a formula for $N_{f2}$ in the presence of residual stress turns out to be not quite as easy as for $N_{f1}$. In order to keep to the most simple approach, we have not taken into account possible changes of residual stress due to micro-yield up to $N_{f1}$ cycles. At $N_{f1}$, however, something will happen to the residual stress. It could change in a variety of ways. Let us consider the most simple case again by pretending that the component which has failed at $N_{f1}$ can stand neither tension nor compression any longer. As a consequence, above $N_{f1}$ residual stress will be gone (an alternative situation is mentioned at the end of this section). Thus it may happen that after failure of one component, the stress on the remaining one is decreased, unlike all possible situations in the absence of residual stress. With $\sigma_{K1} > \sigma_K$ in $N_{Kf}(N_1, \sigma_{K1}, \sigma_K)$, Inequality 9 has to be modified. Instead of Inequality 10, we obtain the relations;

if $\sigma_{K1} < \sigma_K$ then Inequality 10 holds;

if $\sigma_{K1} > \sigma_K$ then

$$N_{Kf}(\sigma_{K1}) < N_{Kf}(N_1, \sigma_{K1}, \sigma_K) < N_{Kf}(\sigma_K)$$
$$\text{and} \quad N_1 < N_{Kf}(N_1, \sigma_{K1}, \sigma_K). \qquad (28)$$

Once again, the stress variables $\sigma_{K1}$ and $\sigma_K$ are understood to be auxiliary quantities constructed by means of Goodman's law in order to replace pairs of variables $\sigma_{K1}^m$, $\sigma_{K1}^a$ and $\sigma_K^m$, $\sigma_K^a$, as indicated in Equation 21. Thus $\sigma_{K1}$ and $\sigma_K$ are always positive.

By modifying Equations 21 for residual stress,

$$\begin{aligned} \sigma_{F1}^m &= \sigma^m E_F/E_I + \sigma_0/v \\ \sigma_{M1}^m &= \sigma^m E_M/E_I - \sigma_0/(1-v) \\ \sigma_{F1}^a = \sigma^a E_F/E_I \quad & \sigma_{M1}^a = \sigma^a E_M/E_I \\ \sigma_F^{m,a} = \sigma^{m,a}/v \quad & \sigma_M^{m,a} = \sigma^{m,a}/(1-v). \end{aligned} \qquad (29)$$

$N_{f2}$ may be written as Inequality 31. Inequalities 30 and 31 reproduce the formulae for the more special situations of zero residual stress or pulsating tension, i.e. $\sigma^a = \sigma^m$.

$$S_1 \equiv \frac{2\sigma_{Mu}\sigma^a E_M}{\sigma_{Mu}E + (\sigma^a - \sigma^m)E_m + \sigma_0 E/(1-v)}, \quad S_3 \equiv \frac{2\sigma_{Mu}\sigma^a}{(1-v)\sigma_{Mu} + \sigma^a - \sigma^m}$$
$$S_2 \equiv \frac{2\sigma_{Fu}\sigma^a E_F}{\sigma_{Fu}E + (\sigma^a - \sigma^m)E_F - \sigma_0 E/v}, \quad S_4 \equiv \frac{2\sigma_{Fu}\sigma^a}{v\sigma_{Fu} + \sigma^a - \sigma^m}$$
$$N_{f1}(\sigma^m, \sigma^a) = \min \begin{cases} N_{Mf}(S_1) & \to \text{matrix fails first} \\ N_{Ff}(S_2) & \to \text{fibre fails first.} \end{cases} \qquad (30)$$

If the matrix fails first:

if $\sigma_0 < v(1-v)\sigma_{Fu}E_M/E$ then

$$N_{Ff}(S_4) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{Ff}(S_2) \text{ and}$$
$$N_{f1}(\sigma^m, \sigma^a) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{f1}(\sigma^m, \sigma^a) + N_{Ff}(S_4)$$

if $\sigma_0 > v(1-v)\sigma_{Fu}E_M/E$ then

$$N_{Ff}(S_2) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{Ff}(S_4) \text{ and}$$
$$N_{f1}(\sigma^m, \sigma^a) \leqslant N_{f2}(\sigma^m, \sigma^a);$$

if the fibre fails first:

if $\sigma_0 > -v(1-v)\sigma_{Mu}E_F/E$ then

$$N_{Mf}(S_3) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{Mf}(S_1) \text{ and}$$

$$N_{f1}(\sigma^m, \sigma^a) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{f1}(\sigma^m, \sigma^a) + N_{Mf}(S_3);$$

if $\sigma_0 < -v(1-v)\sigma_{Mu}E_F/E$ then

$$N_{Mf}(S_1) \leqslant N_{f2}(\sigma^m, \sigma^a) \leqslant N_{Mf}(S_3) \text{ and}$$

$$N_{f1}(\sigma^m, \sigma^a) \leqslant N_{f2}(\sigma^m, \sigma^a). \tag{31}$$

After having found the result by a more formal way of reasoning, we should briefly discuss what happens in reality. Let us consider a composite system, the damage of which is initiated by matrix failure. If the fibres were given some tensile pre-stress, the matrix would fail later than in the case of zero pre-stress, which means that $N_{f1}$ would increase. This could not go on, for increasing fibre stress would imply earlier fibre failure. Obviously there must be an optimum residual stress that causes the two components to suffer damage at the same number of cycles, no strength resources being wasted. This will occur if the two alternative numbers in Equation 30 coincide.

Going back to possible changes of residual stress at $N = N_{f1}$, a different situation should be mentioned briefly. It is evidently possible that the damaged component may still be able to sustain compressional load, therefore the residual stress need not vanish at $N_{f1}$. No additional theory is needed for deriving the corresponding formulae.

## 7. Deriving more general results

Using the formulae derived so far for the computation of numerical results for special systems, one will run into difficulties as soon as one comes across a material whose fatigue curve has not been measured at pulsating tension, but at alternating or fluctuating stress. All that has to be done in this case is to replace the pulsating tension functions to be met in our formulae by equivalent fluctuating stress functions, provided there is some reason to believe that the components in question obey Goodman's law (since alternating stress is a special case of fluctuating stress, the former need not be dealt with separately). The equivalence of $N_{Kf}(\sigma_K)$ and $N_{Kf}(\sigma_K^{m*}, \sigma_K^{a*})$ is met if $\sigma_K^{a*}$ is chosen so as to obey Relation 15;

$$N_{Kf}(\sigma_K) = N_{Kf}\left(\sigma_K^{m*}, \sigma_K \frac{\sigma_{Ku} - \sigma_K^{m*}}{2\sigma_{Ku} - \sigma_K}\right) \tag{32}$$

By means of this formula, our results may be transformed into a shape which would make it possible to use component fatigue curves measured at any $\sigma_K^{m*}$. Transforming Relations 30 and 31 in this way, we obtain our most general formulae for $N_{f1}$ and $N_{f2}$, given by Relations 33 and 34.

$$S_1 \equiv \sigma^a E_M \frac{\sigma_{Mu} - \sigma_M^{m*}}{\sigma_{Mu}E - \sigma^m E_M + \sigma_0 E/(1-v)}, \quad S_3 \equiv \sigma^a \frac{\sigma_{Mu} - \sigma_M^{m*}}{(1-v)\sigma_{Mu} - \sigma^m}$$

$$S_2 \equiv \sigma^a E_F \frac{\sigma_{Fu} - \sigma_F^{m*}}{\sigma_{Fu}E - \sigma^m E_F - \sigma_0 E/v}, \quad S_4 \equiv \sigma^a \frac{\sigma_{Fu} - \sigma_F^{m*}}{v\sigma_{Fu} - \sigma^m}.$$

$$N_{f1}(\sigma^m, \sigma^a) = \min\begin{cases} N_{Mf}(\sigma_M^{m*}, S_1) \\ N_{Ff}(\sigma_F^{m*}, S_2) \end{cases} \rightarrow \begin{array}{l} \text{matrix fails first} \\ \text{fibre fails first.} \end{array} \tag{33}$$

A formula for $N_{f2}$ is obtained by substituting in Relation 31

$$\begin{gathered} N_{Mf}(S_1) \rightarrow N_{Mf}(\sigma_M^{m*}, S_1), \quad N_{Mf}(S_3) \rightarrow N_{Mf}(\sigma_M^{m*}, S_3), \\ M_{Ff}(S_2) \rightarrow N_{Ff}(\sigma_F^{m*}, S_2), \quad N_{Ff}(S_4) \rightarrow N_{Ff}(\sigma_F^{m*}, S_4), \end{gathered} \tag{34}$$

with $S_1, S_2, S_3, S_4$ as defined in this section. Our formalism applies to any monotonically decreasing component fatigue function. Additional restrictions are not required.

Often fatigue functions are held to be approximated sufficiently well by a downward straight line in the $\sigma_{Kf}(\ln N)$ plot, which around $N = 10^7$ suddenly turns into a horizontal straight line. Though this is not always so [6], this type of curve has been chosen for convenience in our plots representing fatigue of the components. It should be mentioned, however, that the assumption of piecewise linear fatigue curves in logarithmic representation is not consistent with Goodman's law, which is evident from Equation 32: If $N_{Kf}(\sigma_K^{m*}, \sigma_K(\sigma_{Ku} - \sigma_K^{m*})/(2\sigma_{Ku} - \sigma_K))$ is linear by experiment in $\sigma_K(\sigma_{Ku} - \sigma_K^{m*})/(2\sigma_{Ku} - \sigma_K)$, it cannot be linear in $\sigma_K$, and vice versa. Fortunately, this inconsistency is not serious as long as $\sigma_K$ is much lower than $2\sigma_{Ku}$, which is guaranteed with the exception of low-cycle fatigue, the latter being omitted here.

It is possible to adapt our formulae for user's special needs, thereby simplifying them. For example, only the so-called fatigue limit may be of interest in some cases, instead of the whole curve, or pulsating tension instead of fluctuating stress. Nowadays widespread use of small computers, into which our most comprehensive formulae may be fed without much difficulty, removes the need for adapting the formulae to special cases. These remarks are not meant to discourage attempts at improving the underlying model by modifying its assumptions.

The case of constant strain amplitude, treated briefly in Section 2 at pulsating tension only, may be considered at more general loading conditions. Equations 22, 30 and 33 are valid in this case, too, but $N_{f2}$ is different. It is still simple as long as residual stress is not considered, otherwise the result will have a structure similar to that of Relation 31.

## 8. Comparison with experiment and discussion of other work

Among reports on the performance of composite materials under fatigue conditions, there is little evidence of residual stress having a great effect. This may be partly due to the fact mentioned above that composites have seldom been tested along with their components, and if done so, usually no special attention has been paid to residual stress.

As an instructive example of an application of our theory, we shall discuss experimental results of a steel wire/Al alloy composite [8] different from those mentioned in Section 4. The 18 Ni maraging steel has been tested by the conventional rotary-bending fatigue method, which is roughly equivalent to an alternating stress test. Thus $N_{Ff}(\sigma_F^{m*}, \sigma_F^{a})$, with $\sigma_F^{m*} = 0$, can be taken as given, which allows us to calculate the composite fatigue curves from Relations 33 and 34. The composite had been tested with pulsating tension, and so we are concerned here with $\sigma^m = \sigma^a = \sigma/2$. The matrix curve is known from [9] at mean stress $\sigma_M^{m*} = 24$ N mm$^{-2}$, so that $N_{f1}$ is given by the smaller of the two numbers

$$N_{f1}(\sigma) = N_{f1}(\sigma/2, \sigma/2) = \min \begin{cases} N_{Mf}\left(\sigma_M^{m*}, \dfrac{\sigma E_M(\sigma_{Mu} - \sigma_M^{m*})}{2E_I(\sigma_{Mu} + \sigma_0/(1-v)) - \sigma E_M}\right) \\ N_{Ff}\left(0, \dfrac{\sigma E_F \sigma_{Fu}}{2E_I(\sigma_{Fu} - \sigma_0/v) - \sigma E_F}\right) \end{cases}$$

With the component parameters $E_M = 70\,000$ N mm$^{-2}$, $E_F = 208\,000$ N mm$^{-2}$, $\sigma_{Mu} = 350$ N mm$^{-2}$, $\sigma_{Fu} = 2800$ N mm$^{-2}$, the component fatigue curves as shown in Fig. 8, $v = 0.18$, and in the absence of residual stress, the first of the two alternative numbers is the smaller one throughout the whole range of $\sigma$. Thus we may state that $N_{f1}(\sigma) = N_{Mf}$ $(24, 326\sigma/(95 - \sigma))$, which indicates that the matrix fails first (all stresses in N mm$^{-2}$). We have only to consider the first alternative in Relation 34. It starts with a condition, which is met in our case, so that all that remains are two restrictions for the number $N_{f2}$,

$$N_{Ff}\left(0, \frac{\sigma\sigma_{Fu}}{2v\sigma_{Fu} - \sigma}\right) < N_{f2}(\sigma) <$$

$$N_{f1}(\sigma) \quad \left(0, \frac{\sigma E_F \sigma_{Fu}}{2E_I(\sigma_{Fu} - \sigma_0/v) - \sigma E_F}\right) \quad \text{and}$$

$$N_{f1}(\sigma) < N_{f2}(\sigma) < N_{Ff} + N_{Ff} \left(0, \frac{\sigma\sigma_{Fu}}{2v\sigma_{Fu} - \sigma}\right)$$

One finds out that $N_{Ff}(0, 2800\sigma/(1000 - \sigma))$ is a very small contribution compared with $N_{f1}(\sigma)$ at any $\sigma$, so that $N_{f2}$ practically coincides with $N_{f1}$. In order to demonstrate how residual stress may affect fatigue, the resulting curve for $\sigma_0 = 100\,\mathrm{N\,mm^{-2}}$ is shown beside that for $\sigma_0 = 0$ in Fig. 8. Once again, the limits for $N_{f2}$ are very narrow so that their width is smaller than that of the lines in our figures, furthermore, $N_{f2}$ coincides with $N_{f1}$ in this case, too.

It should be noted that the distinct angles seen in our curves are an aid to better understanding the method of constructing the curves. Any fine features of the curves, which would be swamped in experimental scatter anyway, are not to be considered as a primary result, but as a mere by-product of the theory. It is not known whether the samples of Fig. 8 did really change their residual stress around $N \approx 10^5$, as is suggested by the graphical representation.

Another approach to the fatigue strength of composites, which is based on a hypothesis by Tavernelli and Coffin [10], has been used by Baker *et al.* in a series of papers (see, for instance, [11]). The essence of the approach is to calculate the

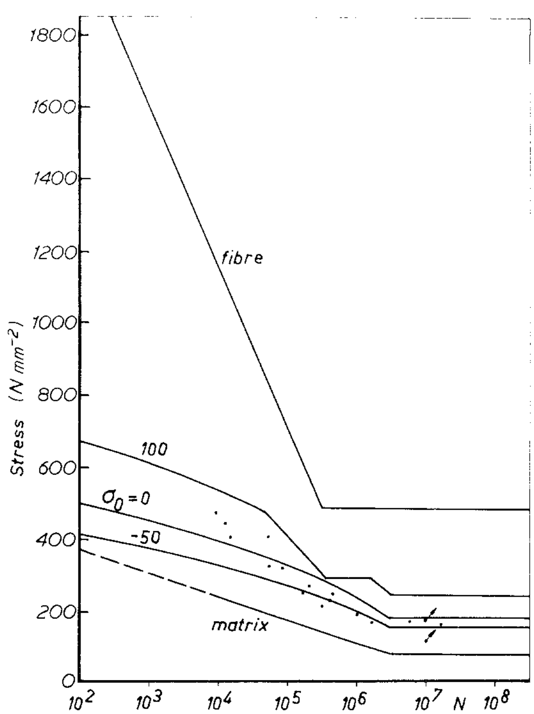


*Figure 8* Fibre: rotational bending fatigue curve of 18Ni-maraging steel wire. Matrix: fluctuating stress fatigue curve of AlMgSi at the mean stress of $\sigma_M^{m*} = 24\,\mathrm{N\,mm^{-2}}$. Derived pulsating tension curves of the composites at three different levels of residual stress are shown, dots representing experimental data.

plastic strain which the matrix is subjected to at every cycle. This strain is held to be inversely proportional to the square root of $N_{Mf}$. Thus the approach entails the calculation of matrix fatigue behaviour. If the component fatigue curves are known, however, one should not dispense with this information, but use it for the construction of the composite fatigue curves, which is done in our approach. The authors [11] discuss various mechanisms causing fatigue failure of the composite, including interfacial failure of several types as well as shear stress concentrations at fibre ends. Though phenomena of this kind may be essential in the line of fatigue, they are considered to be beyond the scope of a first quantitative approach. Thus they are given no special attention in this paper. There is no shear at the interfaces of continuous fibre composites with tensile load, except near the ends or at imperfections.

A rather peculiar failure mode has been reported in a monograph by Ivanova *et al.* [12], where broken samples are shown with the fibres pulled out unbroken at full length. This phenomenon of complete debonding must have been brought about by interface shear stress. According to the above statement, shear stress at the fibre end or at a matrix fatigue crack could have been the initial cause of debonding. In the latter case, failure is described by our $\sigma_{f1}$ (or $N_{f1}$), but there is no $\sigma_{f2}$. Thus it depends on whether the interface crack had run from within the sample towards its end or conversely, whether our theory applies to this case.

A formula corresponding to our Equation 3b is given by [12]. This formula has been modified by Tamayo [13] in order to take into account residual stress. He simply replaced $\sigma_{Mf}$ by $(\sigma_{Mf} + \sigma_{M0})$ in Equation 3b, thereby clearly over-estimating the effect of residual stress. Tamayo treats the phenomenon mathematically as if residual stress would change the upper matrix stress only, leaving the lower matrix stress at zero in the pulsating tension test. If the composite is tested at pulsating tension, i.e. with zero lower stress, the components, of course, are subjected to fluctuating stress as soon as residual stress comes into play. This has been taken into account in our approach.

As a means of graphical representation of the efficiency of fibres in improving fatigue properties of the composite, the fatigue strength at a certain number of cycles may be plotted versus volume

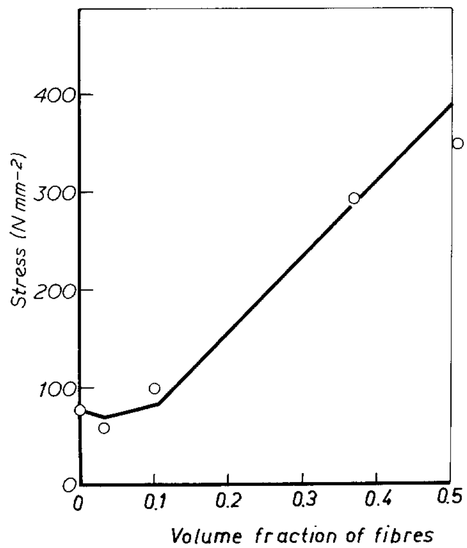


*Figure 9* Pulsating tension fatigue strength of Cu/W composites versus volume fraction, derived from component data by means of Relation 31. Dots represent experimental data as in [14].

fraction of fibres. In another monograph [14], Ivanova *et al.* plotted the pulsating tension strength of Cu/W composites at $N = 10^7$. With the component properties $E_M = 100\,000\,\mathrm{N\,mm^{-2}}$, $E_F = 350\,000\,\mathrm{N mm^{-2}}$, $\sigma_{Mu} = 300\,\mathrm{N mm^{-2}}$, $\sigma_{Fu} = 3000\,\mathrm{N\,mm^{-2}}$, $\sigma_{Mf}(10^7) = 80\,\mathrm{N\,mm^{-2}}$, $\sigma_{Ff}(10^7) = 800\,\mathrm{N mm^{-2}}$, the latter value being assumed, our Relation 31 provides a curve which is shown along with the experimental data in Fig. 9. The downward slope of the first section of the curve is due to thermal stress induced by cooling of the sample after hot pressing. Thermal stress has been held to be limited to about $50\,\mathrm{N\,mm^{-2}}$, since larger stress would relax soon in the recrystallized Cu matrix.

The few examples of fairly good coincidence of experiment and theory may serve as evidence indicating that fatigue of some kinds of composites is really governed by the simple mechanism which our model is based on. Very probably there are other kinds of composites that behave under cyclic load in a more involved manner. In such cases our approach may still serve as a means for separating synergetic effects from those of the simplest approximation.

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